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Curvature properties of compact spacelike hypersurfaces in de Sitter space[1]

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Abstract: In this paper we establish a sufficient condition for a compact spacelike hypersurface in de Sitter space to be spherical in terms of a lower bound for the square of its mean curvature. Our result will be a consequence of the maximum principle for the Laplacian operator. We also derive some other applications and consequences of our main result. In particular, we establish another sufficient condition for a compact spacelike hypersurface in de Sitter space to be spherical in terms of a pinching condition for its scalar curvature, as well as in terms of the Ricci curvature and in terms of the higher order mean curvatures.

Keywords: de Sitter space, spacelike hypersurface, mean curvature, scalar curvature, Ricci curvature, higher order mean curvature, hyperbolic image.

MS classification: 53C42; 53B30, 53C50.

1. Introduction and statement of the main results

Let $\mathbf{L}^{n+2}$ be the $(n+2)$-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric tensor $\langle\cdot\,,\cdot\rangle$ given by

$$\langle\cdot\,,\cdot\rangle = (dx_1)^2 + \cdots + (dx_{n+1})^2 - (dx_{n+2})^2,$$

where $(x_1, \ldots, x_{n+2})$ are the canonical coordinates of $\mathbb{R}^{n+2}$. The $(n+1)$-dimensional unitary de Sitter space is given as the following hyperquadric of $\mathbf{L}^{n+2}$,

$$\mathbf{S}_1^{n+1} = \{x \in \mathbf{L}^{n+2} : \langle x, x\rangle = 1\}.$$

As it is well known, $\mathbf{S}_1^{n+1}$ inherits from $\mathbf{L}^{n+2}$ a Lorentzian metric which makes it the standard model of a Lorentzian space of constant sectional curvature one. A smooth immersion

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$\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ of an n-dimensional connected manifold M is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on M, which, as usual, is also denoted by $\langle \cdot , \cdot \rangle$.

The study of spacelike hypersurfaces in de Sitter space $\mathbf{S}_1^{n+1}$ has been of increasing interest in the last years, because of their nice Bernstein-type properties. Since Goddard [6] conjectured in 1977 that the only complete spacelike hypersurfaces in $\mathbf{S}_1^{n+1}$ with constant mean curvature should be the totally umbilical ones, which turned out to be false in this original statement, an important number of authors have considered the problem of characterizing the totally umbilical spacelike hypersurfaces of de Sitter space in terms of some appropriate geometric assumptions. Actually, Akutagawa [1] showed that Goddard's conjecture is true if the constant mean curvature H satisfies $0 \leqslant H^2 \leqslant 1$ when $n = 2$, and $0 \leqslant H^2 < 4(n-1)/n^2$ when $n \geqslant 3$. On the other hand, Montiel proved in [10] that Goddard's conjecture is also true under the additional hypothesis of the compactness of the hypersurface. We also refer to [13] for an alternative proof of both facts given by Ramanathan in the 2-dimensional case. More recently, Cheng and Ishikawa [4] have shown that the totally umbilical round spheres are the only compact spacelike hypersurfaces in de Sitter space with constant scalar curvature $S < n(n-1)$. Li [8] and Zheng [15, 16] also obtained interesting characterizations of these hypersurfaces under the hypothesis of constant scalar curvature. In [2] the authors, jointly with Romero, have recently found some other characterizations of the totally umbilical round spheres in de Sitter space as the only compact spacelike hypersurfaces with constant higher order mean curvature, under appropriate hypothesis. In another direction, Oliker investigated in [12] the stability of the Bernstein-type property relative to perturbation of the data, proving that if the mean curvature of a complete spacelike hypersurface in $\mathbf{S}_1^{n+1}$ is such that $H^2 < 4(n-1)/n^2$ and H is close to a constant, then the hypersurface is close to be a totally umbilical round sphere.

In this paper we will characterize the totally umbilical round spheres of de Sitter space in terms of some appropriate bounding conditions for their curvatures. Let us recall that the Gauss map of a spacelike hypersurface in de Sitter space can be regarded as a map $N : M \longrightarrow \mathbf{H}^{n+1}$, where $\mathbf{H}^{n+1} \subset \mathbf{L}^{n+2}$ denotes the $(n+1)$-dimensional hyperbolic space, that is

$$\mathbf{H}^{n+1} = \{x \in \mathbf{L}^{n+2} : \langle x, x \rangle = -1\}.$$

The image $N(M)$ will be called the *hyperbolic image* of the hypersurface.

In [3], the second named author has recently shown that if the hyperbolic image of a compact spacelike hypersurface M is contained in a geodesic ball in $\mathbf{H}^{n+1}$ of radius $\varrho > 0$ and the Ricci curvature of M satisfies the following pinching condition

$$\mathrm{Ric} \leqslant \frac{n-1}{\cosh^2(\varrho)},$$

then M must be a totally umbilical round sphere of radius $\cosh(\varrho)$. The proof of this result in [3] was a consequence of an integral formula involving the Ricci curvature and the scalar curvature of the hypersurface. We will establish here a sufficient condition for a compact spacelike hypersurface in de Sitter space to be spherical in terms of a lower bound for the square of its mean curvature. Our proof will be a consequence of the maximum principle for the Laplacian operator. Specifically we will prove the following result.

Theorem 1. *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space such that its hyperbolic image is contained in a geodesic ball in $\mathbf{H}^{n+1}$ of radius $\varrho \geqslant 0$. If the mean curvature H of M satisfies*

$$H^2 \geqslant \tanh^2(\varrho),$$

then M must be a totally umbilical round sphere of radius $\cosh(\varrho)$.

As an application of this we obtain the following improvement of the theorem in [3].

Theorem 2. *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space such that its hyperbolic image is contained in a geodesic ball in $\mathbf{H}^{n+1}$ of radius $\varrho \geqslant 0$. If the scalar curvature S of M satisfies*

$$S \leqslant \frac{n(n-1)}{\cosh^2(\varrho)},$$

then M must be a totally umbilical round sphere of radius $\cosh(\varrho)$.

Actually, our Theorem 1 will be obtained as a consequence of the following stronger result.

Theorem 3. *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space, $a \in \mathbf{L}^{n+2}$ a timelike unit vector and $r \geqslant 0$, verifying that $\langle a, \psi\rangle^2 \leqslant \sinh^2(r)$. If the mean curvature H of M satisfies*

$$H^2 \geqslant \tanh^2(r),$$

then M must be a totally umbilical round sphere of radius $\cosh(r)$.

In other words, if $\psi(M) \subset \mathbf{S}_1^{n+1}$ is contained in the timelike bounded region

$$B(a, r) = \left\{x \in \mathbf{S}_1^{n+1} : -\sinh(r) \leqslant \langle a, x\rangle \leqslant \sinh(r)\right\} \subset \mathbf{S}_1^{n+1},$$

whose boundaries are two totally umbilical round spheres of radius $\cosh(r) \geqslant 1$ and constant mean curvature $h^2(r) = \tanh^2(r)$, and the mean curvature of M satisfies $H^2 \geqslant h^2(r)$, then M must be one of these two round spheres. We will refer to r as the *timelike radius* of the region $B(a, r) \subset \mathbf{S}_1^{n+1}$. Similar results of this nature for compact hypersurfaces bounded in a geodesic ball of a Riemannian space were given by Markvorsen in [9], where he generalized the classical result of Koutroufiotis [7] on surfaces in Euclidean space $\mathbf{E}^3$.

2. Preliminaries

Throughout this paper we will deal with compact spacelike hypersurfaces in de Sitter space. Observe that every such hypersurface $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ is diffeomorphic to an n-sphere by means of the map $F = \Pi \circ \phi \circ \psi : M \longrightarrow \mathbf{S}^n$, where $\Pi : \mathbf{S}^n \times \mathbb{R} \longrightarrow \mathbf{S}^n$ is the projection onto $\mathbf{S}^n$ and $\phi^{-1} : \mathbf{S}^n \times \mathbb{R} \longrightarrow \mathbf{S}_1^{n+1}$ is given by $\phi^{-1}(u, v) = \left(\left(\sqrt{1+v^2}\right)u, v\right)$. Indeed, F is a local diffeomorphism, and the compactness of M and the simply connectedness of $\mathbf{S}^n$ imply that F is a global one. In particular, every compact spacelike hypersurface in de Sitter space is orientable and there exists a timelike unit normal field N globally defined on M. We will refer to N as the *Gauss map* of the immersion and we will say that M is oriented by N.

Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be an immersed compact spacelike hypersurface in de Sitter space, and let N be its Gauss map. In order to set up the notation to be used later, we will denote by ∇^{o}, $\bar{\nabla}$ and ∇ the Levi-Civita connections of $\mathbf{L}^{n+2}$, $\mathbf{S}_1^{n+1}$ and M, respectively. Then the Gauss and Weingarten formulas for M in $\mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ are given respectively by

$$\nabla^{\mathrm{o}}{}_X Y = \bar{\nabla}_X Y - \langle X, Y\rangle \psi = \nabla_X Y - \langle AX, Y\rangle N - \langle X, Y\rangle \psi, \tag{1}$$

and

$$A(X) = -\bar{\nabla}_X N = -\nabla^{\mathrm{o}}_X N, \tag{2}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$. Here $A : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ stands for the shape operator of M in $\mathbf{S}_1^{n+1}$ associated to N, and $H = -(1/n)\,\mathrm{tr}(A)$ defines the mean curvature of the hypersurface.

Before establishing the main results, we will briefly study the totally umbilical round spheres of de Sitter space.

Example 4. Let $a \in \mathbf{L}^{n+2}$ be a fixed timelike unit vector, $\langle a, a\rangle = -1$, and let $f_a : \mathbf{S}_1^{n+1} \longrightarrow \mathbb{R}$ be defined by $f_a(x) = \langle a, x\rangle$. For a given $r \in \mathbb{R}$, let $\mathbf{S}^n(a, r) = f_a^{-1}(\sinh(r))$. Observe that the gradient (in de Sitter space) of f_a is

$$\bar{\nabla} f_a(x) = a - \langle a, x\rangle x, \quad x \in \mathbf{S}_1^{n+1},$$

so that

$$\langle \bar{\nabla} f_a(x), \bar{\nabla} f_a(x)\rangle = -\cosh^2(r) < 0, \quad x \in \mathbf{S}^n(a, r).$$

Therefore, $\mathbf{S}^n(a, r)$ is a spacelike hypersurface embedded in de Sitter space with Gauss map

$$N(x) = \frac{1}{\cosh(r)}(a - \sinh(r)x),$$

and shape operator

$$A(X) = -\nabla^{\mathrm{o}}_X N = \tanh(r)X.$$

In particular, $\mathbf{S}^n(a, r)$ is a totally umbilical hypersurface with constant mean curvature

$$h^2(r) = \tanh^2(r),$$

and its hyperbolic image is the geodesic sphere in $\mathbf{H}^{n+1}$ of radius r centered at a,

$$N(\mathbf{S}^n(a, r)) = \{q \in \mathbf{H}^{n+1} : \langle a, q\rangle = -\cosh(r)\}.$$

Even more, it is not difficult to see that $\mathbf{S}^n(a, r)$ is a round sphere of radius $\cosh(r)$ contained in the spacelike hyperplane orthogonal to a determined by $\langle a, x\rangle = \sinh(r)$. We will refer to $\mathbf{S}^n(a, r)$ as the *totally umbilical round spheres* of de Sitter space. Finally, let us remark that they are all the compact umbilical spacelike hypersurfaces in $\mathbf{S}_1^{n+1}$.

3. First results

Let $a \in \mathbf{L}^{n+2}$ be a fixed arbitrary vector, and consider the height function $\langle a, \psi\rangle$ on M. From (1) it is easy to see that its gradient is given by

$$\nabla\langle a, \psi\rangle = a^T = a + \langle a, N\rangle N - \langle a, \psi\rangle\psi, \tag{3}$$

where a^T is tangent to M. Thus

$$\langle a, N\rangle^2 = -\langle a, a\rangle + \langle a, \psi\rangle^2 + |\nabla\langle a, \psi\rangle|^2 \geqslant -\langle a, a\rangle + \langle a, \psi\rangle^2, \tag{4}$$

with equality at the critical points of $\langle a, \psi\rangle$. By taking covariant derivative in (3) and using (1) and (2), we obtain from $\nabla^o a = 0$ that

$$\nabla_X a^T = -\langle a, N\rangle A(X) - \langle a, \psi\rangle X$$

for $X \in \mathcal{X}(M)$. Therefore, the Hessian of $\langle a, \psi\rangle$ is given by

$$\nabla^2\langle a, \psi\rangle(X, Y) = \langle\nabla_X a^T, Y\rangle = -\langle a, N\rangle\langle AX, Y\rangle - \langle a, \psi\rangle\langle X, Y\rangle, \tag{5}$$

for $X, Y \in \mathcal{X}(M)$, and its Laplacian is

$$\Delta\langle a, \psi\rangle = -\langle a, N\rangle\,\mathrm{tr}(A) - n\langle a, \psi\rangle = nH\langle a, N\rangle - n\langle a, \psi\rangle. \tag{6}$$

Lemma 5. *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space, let $a \in \mathbf{L}^{n+2}$ be a fixed timelike unit vector and let $r_1 < r_2$ be real numbers for which $\psi(M)$ is contained in the region*

$$\{x \in \mathbf{S}_1^{n+1} : -\sinh(r_2) \leqslant \langle x, a\rangle \leqslant -\sinh(r_1)\}.$$

Let us orient M by the timelike unit normal field N which is in the same time-orientation as a. Then the mean curvature H of M satisfies

$$\min H \leqslant \tanh(r_2) \quad \text{and} \quad \max H \geqslant \tanh(r_1).$$

Proof. Let us choose $p_{\max} \in M$ a point where the function $\langle a, \psi\rangle$ attains its maximum over M. Since $\langle a, N\rangle \leqslant -1 < 0$, it follows from (4) that

$$\langle a, N(p_{\max})\rangle = -\sqrt{1 + \langle a, \psi(p_{\max})\rangle^2},$$

so that using also (6) we get

$$\begin{aligned}&\Delta\langle a, \psi\rangle(p_{\max})\\ &\quad = -n\Big(H(p_{\max})\sqrt{1 + \langle a, \psi(p_{\max})\rangle^2} + \langle a, \psi(p_{\max})\rangle\Big) \leqslant 0,\end{aligned}$$

that is,

$$H(p_{\max}) \geqslant \frac{-\langle a, \psi(p_{\max})\rangle}{\sqrt{1 + \langle a, \psi(p_{\max})\rangle^2}}.$$

Since the function $t/\sqrt{1+t^2}$ is increasing on $\mathbb{R}$ and $-\langle a, \psi(p_{\max})\rangle \geqslant \sinh(r_1)$, we finally obtain that

$$\max H \geqslant H(p_{\max}) \geqslant \frac{\sinh(r_1)}{\sqrt{1+\sinh^2(r_1)}} = \tanh(r_1).$$

Analogously, working at a point $p_{\min} \in M$ where the function $\langle a, \psi\rangle$ attains its minimum, we obtain that

$$\min H \leqslant H(p_{\min}) \leqslant \frac{-\langle a, \psi(p_{\min})\rangle}{\sqrt{1+\langle a, \psi(p_{\min})\rangle^2}} \leqslant \tanh(r_2). \quad \square$$

In order to state a first result related to our Theorem 3, let us introduce the following terminology. Let $a \in \mathbf{L}^{n+2}$ be a timelike unit vector. As we have seen in Example 4, the level set given by $\{x \in \mathbf{S}_1^{n+1} : \langle a, x\rangle = 0\}$ defines a round sphere of radius one which is a totally geodesic hypersurface in $\mathbf{S}_1^{n+1}$. We will refer to that sphere as the *equator* of $\mathbf{S}_1^{n+1}$ determined by a. This equator divides the de Sitter space into two connected components, the *chronological future* (with respect to the time-orientation determined by a), which is given by

$$B^+(a) = \{x \in \mathbf{S}_1^{n+1} : \langle a, x\rangle < 0\},$$

and the *chronological past*, given by

$$B^-(a) = \{x \in \mathbf{S}_1^{n+1} : \langle a, x\rangle > 0\}.$$

Proposition 6. *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space and let $a \in \mathbf{L}^{n+2}$ be a fixed timelike unit vector such that $\psi(M)$ is contained in the chronological future of the equator determined by a, or in its chronological past. If the mean curvature H of M satisfies*

$$H^2 \geqslant \frac{\langle \psi, a\rangle^2}{1+\langle \psi, a\rangle^2}$$

on M, then M must be a totally umbilical round sphere.

Proof. Let us assume, for instance, that $\psi(M)$ is contained in the chronological future; the case of the chronological past is similar. Let us choose on M the orientation given by the timelike unit normal field N which is in the same time-orientation that a, so that $\langle a, N\rangle \leqslant -1 < 0$. From the hypothesis on H it follows that either

$$H \geqslant \frac{-\langle a, \psi\rangle}{\sqrt{1+\langle a, \psi\rangle^2}} > 0, \tag{7}$$

or

$$H \leqslant \frac{\langle a, \psi\rangle}{\sqrt{1+\langle a, \psi\rangle^2}} < 0. \tag{8}$$

Let us first see that (8) cannot occur. Indeed, if (8) holds then from (6) we obtain that $\Delta\langle a, \psi\rangle > 0$ on M, which is not possible from divergence theorem. On the other hand, using (4), equation

(7) implies that

$$\Delta\langle a, \psi\rangle \leqslant -n\langle a, \psi\rangle\left(1 + \frac{\langle a, N\rangle}{\sqrt{1+\langle a, \psi\rangle^2}}\right) \leqslant 0.$$

Using now Hopf–Bochner theorem, we conclude that the function $\langle a, \psi\rangle$ is constant on M, that is,

$$\psi(M) = \{x \in \mathbf{S}_1^{n+1} : \langle a, x\rangle = \sinh(r)\}$$

for a negative real constant $r < 0$, and therefore M is a totally umbilical round sphere of radius $\cosh(r)$ (Example 4). $\square$

As a consequence of Proposition 6 we obtain the following partial result of our Theorem 3.

Corollary 7. *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space such that $\psi(M)$ is contained in the bounded chronological future region*

$$B^+(a, r) = \{x \in \mathbf{S}_1^{n+1} : -\sinh(r) \leqslant \langle x, a\rangle < 0\}$$

or in the bounded chronological past region

$$B^-(a, r) = \{x \in \mathbf{S}_1^{n+1} : 0 < \langle x, a\rangle \leqslant \sinh(r)\}$$

for some fixed timelike unit vector $a \in \mathbf{L}^{n+2}$ and $r > 0$. If the mean curvature H of M satisfies $H^2 \geqslant \tanh^2(r)$, then M must be a totally umbilical round sphere of radius $\cosh(r)$.

Proof. Let us assume, for instance, that the hypersurface is contained in $B^+(a, r)$; the case of the chronological past is similar. As in the proof of Proposition 6, let us choose on M the orientation given by the timelike unit normal field N which is in the same time-orientation that a. First of all, observe that using Lemma 5 we get that

$$\min H \leqslant \tanh(r_0) \leqslant \tanh(r) \quad \text{and} \quad \max H > 0, \tag{9}$$

where $0 < r_0 \leqslant r$ is given by $-\sinh(r_0) = \min_{p\in M} \langle a, \psi(p)\rangle$. On the other hand, from the hypothesis on H we know that either

$$H \geqslant \tanh(r) \quad \text{or} \quad H \leqslant -\tanh(r) < 0. \tag{10}$$

From (9) and (10), it follows that necessarily $H \geqslant \tanh(r)$ and moreover $r_0 = r$. Using again that the function $t/\sqrt{1+t^2}$ is increasing on $\mathbb{R}$ and $-\sinh(r) \leqslant \langle a, \psi\rangle$, we have that

$$-\tanh(r) \leqslant \frac{\langle a, \psi\rangle}{\sqrt{1+\langle a, \psi\rangle^2}} < 0$$

and

$$\tanh^2(r) \geqslant \frac{\langle a, \psi\rangle^2}{1+\langle a, \psi\rangle^2}.$$

Proposition 6 allows us to conclude that M is the totally umbilical round sphere of radius $\cosh(r)$ given by $\langle a, \psi\rangle = -\sinh(r)$. $\square$

4. Proof of the main results

The proof of Theorem 3 is based on the well-known maximum principle for the Laplacian operator of a Riemannian manifold.

Maximum principle. *Let f be a smooth function on a Riemannian manifold M, which attains a maximum (respectively, minimum) at a point $p \in M$. If $\mathcal{U}$ is a neighbourhood of p where $\Delta f \geqslant 0$ (respectively, $\Delta f \leqslant 0$), then f is constant on $\mathcal{U}$.*

Proof of Theorem 3. Note that the case $r = 0$ is trivial. Moreover, in Corollary 7 we have already proved the result when $\psi(M)$ is contained in the chronological future or in the chronological past of the equator determined by a. Our objective is to see that if $r \neq 0$, then $\psi(M)$ is necessarily contained in the chronological future or past of the equator. Actually, if this is not the case then there exist non-negative real numbers r_1, r_2, not both vanishing, such that

$$-r \leqslant -r_1 \leqslant 0 \leqslant r_2 \leqslant r,$$

and there exist points $p_{\max}, p_{\min} \in M$ satisfying

$$\max_{p \in M} \langle a, \psi(p)\rangle = \langle a, \psi(p_{\max})\rangle = \sinh(r_1)$$

$$\min_{p \in M} \langle a, \psi(p)\rangle = \langle a, \psi(p_{\min})\rangle = -\sinh(r_2).$$

Therefore, from Lemma 5 we know that

$$\min H \leqslant \tanh(r_2) \tag{11}$$

and

$$\max H \geqslant -\tanh(r_1). \tag{12}$$

On the other hand, the hypothesis on H implies that either

$$H \geqslant \tanh(r) \tag{13}$$

or

$$H \leqslant -\tanh(r). \tag{14}$$

We may assume without loss of generality that $r_1 \geqslant r_2$, that is,

$$r \geqslant r_1 \geqslant r_2 \geqslant 0$$

and moreover $r_1 > 0$. Otherwise, just replace a by $-a$. If (13) holds, then from equation (11) it follows that $r_2 = r = r_1 > 0$. Since $\langle a, \psi(p_{\min})\rangle = -\sinh(r) < 0$, then there exists a neighbourhood $\mathcal{U}$ of $p_{\min}$ where $\langle a, \psi\rangle$ is negative. Using that $-\sinh(r) \leqslant \langle a, \psi\rangle$ on M, at each point $p \in M$ we have from (13)

$$H(p) \geqslant \tanh(r) \geqslant \frac{-\langle a, \psi(p)\rangle}{\sqrt{1 + \langle a, \psi(p)\rangle^2}},$$

which is strictly positive on $\mathcal{U}$. Since $-\langle a, N\rangle \geqslant \sqrt{1+\langle a,\psi\rangle^2}$ on M (see equation (4)), then

$$\Delta\langle a,\psi\rangle \leqslant -n\langle a,\psi\rangle\left(1+\frac{\langle a, N\rangle}{\sqrt{1+\langle a,\psi\rangle^2}}\right) \leqslant 0 \quad \text{on } \mathcal{U}.$$

Therefore, by the maximum principle, $\langle a,\psi\rangle$ is constant on $\mathcal{U}$. Since M is connected, this implies that $\langle a,\psi\rangle$ is constant on M, in contradiction with the fact that $\min_{p\in M}\langle a,\psi(p)\rangle = -\max_{p\in M}\langle a,\psi(p)\rangle < 0$.

On the other hand, if (14) holds, then from (12) it follows that $r_1 = r$. Moreover, since $\langle a,\psi(p_{\max})\rangle = \sinh(r) > 0$, there exist a neighbourhood $\mathcal{V}$ of $p_{\max}$ where $\langle a,\psi\rangle$ is positive. Using that $\langle a,\psi\rangle \leqslant \sinh(r)$ on M, we conclude from (14) that

$$H \leqslant -\tanh(r) \leqslant \frac{-\langle a,\psi\rangle}{\sqrt{1+\langle a,\psi\rangle^2}} < 0 \quad \text{on } \mathcal{V}.$$

Reasoning as above we obtain now that $\Delta\langle a,\psi\rangle \geqslant 0$ on $\mathcal{V}$, and that the function $\langle a,\psi\rangle$ is constant on M and equal to $\sinh(r) > 0$, in contradiction to the fact that its minimum is non-positive. This finishes the proof of Theorem 3. □

Proof of Theorem 1. Let us assume that the hyperbolic image of M is contained in a geodesic ball $\tilde{B}(a,\varrho)$ in $\mathbf{H}^{n+1}$ of radius $\varrho \geqslant 0$ centered at $a \in \mathbf{H}^{n+1}$, $\langle a, a\rangle = -1$. Recall that

$$\tilde{B}(a,\varrho) = \{q \in \mathbf{H}^{n+1} : 1 \leqslant \langle a, q\rangle^2 \leqslant \cosh^2(\varrho)\},$$

so that $\langle a, N(p)\rangle^2 \leqslant \cosh^2(\varrho)$ at each $p \in M$, and from (4)

$$\langle a,\psi\rangle^2 \leqslant \langle a, N\rangle^2 - 1 \leqslant \sinh^2(\varrho)$$

on M. Thus, we are under the hypothesis of Theorem 3. □

Proof of Theorem 2. The Ricci curvature of M is given by

$$\mathrm{Ric}(X,Y) = (n-1)\langle X,Y\rangle - \mathrm{tr}(A)\langle A(X),Y\rangle + \langle A(X), A(Y)\rangle,$$

for $X, Y \in \mathfrak{X}(M)$, so that its scalar curvature is

$$S = \mathrm{tr}(\mathrm{Ric}) = n(n-1) - n^2H^2 + \mathrm{tr}(A^2).$$

Using now the Cauchy–Schwarz inequality we have that $\mathrm{tr}(A^2) \geqslant nH^2$, so that

$$S \geqslant n(n-1)(1-H^2). \tag{15}$$

Thus, if $S \leqslant n(n-1)/\cosh^2(\varrho)$ then $H^2 \geqslant \tanh^2(\varrho)$ and we can apply Theorem 1. □

5. Applications

Theorem 2 can be actually seen as a consequence of the following stronger fact, which follows from inequality (15) and Theorem 3.

Theorem 8. *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space, $a \in \mathbf{L}^{n+2}$ a timelike unit vector and $r \geqslant 0$, verifying that $\langle a, \psi\rangle^2 \leqslant \sinh^2(r)$. If the scalar curvature S of M satisfies*

$$S \leqslant \frac{n(n-1)}{\cosh^2(r)},$$

then M must be a totally umbilical round sphere of radius $\cosh(r)$.

That is, if $\psi(M) \subset \mathbf{S}_1^{n+1}$ is contained in the timelike bounded region

$$B(a, r) = \{x \in \mathbf{S}_1^{n+1} : -\sinh(r) \leqslant \langle a, x\rangle \leqslant \sinh(r)\} \subset \mathbf{S}_1^{n+1},$$

whose boundaries are two totally umbilical round spheres of radius $\cosh(r)$ and constant scalar curvature $s(r) = n(n-1)/\cosh^2(r)$, and the scalar curvature of M satisfies $S \leqslant s(r)$, then M must be one of these two round spheres.

As stated in the Introduction, Markvorsen in [9] obtained results of the same nature of our Theorem 3 for the case of compact hypersurfaces bounded in a geodesic ball of a Riemannian space, generalizing the classical result of Koutroufiotis [7] on surfaces in Euclidean space $\mathbf{E}^3$. This classical result states that if the mean curvature of a compact surface in $\mathbf{E}^3$ satisfies $H^2 \leqslant 1/r^2, r > 0$, then the smallest round ball containing the surface has radius larger than r, unless the surface is a round sphere of radius r. In this terms, our Theorem 3 can be also stated as follows.

Corollary 9. *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space with mean curvature*

$$H^2 \geqslant \tanh^2(r), \quad r \geqslant 0.$$

Then for any timelike direction a, the smallest region $B(a, R)$ containing $\psi(M)$ has timelike radius larger than r, unless the hypersurface is a totally umbilical round sphere of radius $\cosh(r)$.

It is worth pointing out that the inequality in our result, $H^2 \geqslant \tanh^2(r)$, is the opposite to the inequality in the Euclidean case $H^2 \leqslant 1/r^2$, or more generally in the Riemannian case. This is natural because, in contrast to the Riemannian case, when the radius of the round spheres in de Sitter space increases, the square of the mean curvature also increases. As for the scalar curvature of the hypersurface, Theorem 8 becomes as follows.

Corollary 10. *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space with scalar curvature*

$$S \leqslant \frac{n(n-1)}{\cosh^2(r)}, \quad r \geqslant 0.$$

Then for any timelike direction a, the smallest region $B(a, R)$ containing $\psi(M)$ has timelike radius larger than r, unless the hypersurface is a totally umbilical round sphere of radius $\cosh(r)$.

In terms of the hyperbolic image of the hypersurface, our Theorems 1 and 2 can be also stated in the following way.

Corollary 11. *Let* $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ *be a compact spacelike hypersurface in de Sitter space with mean curvature*

$$H^2 \geqslant \tanh^2(\varrho), \quad \varrho \geqslant 0.$$

Then the smallest geodesic ball in $\mathbf{H}^{n+1}$ *containing the hyperbolic image of* M *has radius larger than* ϱ*, unless the hypersurface is a totally umbilical round sphere of radius* $\cosh(\varrho)$*.*

Corollary 12. *Let* $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ *be a compact spacelike hypersurface in de Sitter space with scalar curvature*

$$S \leqslant \frac{n(n-1)}{\cosh^2(\varrho)}, \quad \varrho \geqslant 0.$$

Then the smallest geodesic ball in $\mathbf{H}^{n+1}$ *containing the hyperbolic image of* M *has radius larger than* ϱ*, unless the hypersurface is a totally umbilical round sphere of radius* $\cosh(\varrho)$*.*

Finally, we will also derive some further applications for the case of the higher order mean curvatures of the hypersurface. Let us recall that associated to the shape operator of M there are n algebraic invariants, which are the elementary symmetric functions σ_j of its principal curvatures $\kappa_1, \ldots, \kappa_n$, given by

$$\sigma_j(\kappa_1, \ldots, \kappa_n) = \sum_{i_1 < \cdots < i_j} \kappa_{i_1} \cdots \kappa_{i_j}, \quad 1 \leqslant j \leqslant n.$$

The *jth mean curvature H_j* of the spacelike hypersurface is then defined by

$$\frac{n!}{(n-j)!\, j!} H_j = (-1)^j \sigma_j(\kappa_1, \ldots, \kappa_n) = \sigma_j(-\kappa_1, \ldots, -\kappa_n).$$

When $j = 1$, $H_1 = -(1/n)\operatorname{tr}(A) = H$ is just the mean curvature of M. On the other hand, when $j = n$, $H_n = (-1)^n \det(A)$ defines the Gauss–Kronecker curvature of the spacelike hypersurface, and for $j = 2$, H_2 is, up to a constant, the scalar curvature S of M,

$$S = n(n-1) - \operatorname{tr}(A)^2 + \operatorname{tr}(A^2) = n(n-1)(1 - H_2).$$

In general, it follows from the Gauss equation of the hypersurface that when j is odd H_j is extrinsic and its sign depends on the chosen orientation, while when j is even H_j is intrinsic. We refer the reader to [2] for the details. For the general case of the j-th mean curvature we are able to prove the following results. Similar results for compact hypersurfaces contained in a geodesic ball of a Riemannian space form have been recently given by Vlachos in [14].

Proposition 13. *Let* $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ *be a compact spacelike hypersurface in de Sitter space,* $a \in \mathbf{L}^{n+2}$ *a timelike unit vector and* $r > 0$*, verifying that* $\langle a, \psi\rangle^2 \leqslant \sinh^2(r)$*. If there exists a point* $p_0 \in M$ *where all the principal curvatures* $\kappa_i(p_0)$ *have the same sign, and* *the jth mean curvature H_j of M satisfies*

$$|H_j| \geqslant \tanh^j(r), \quad \text{if } j \text{ is odd}$$

or

$$H_j \geqslant \tanh^j(r), \quad \text{if } j \text{ is even},$$

then M must be a totally umbilical round sphere of radius $\cosh(r)$.

In other words, if there exists a point $p_0 \in M$ where all the principal curvatures $\kappa_i(p_0)$ have the same sign, and $\psi(M) \subset \mathbf{S}_1^{n+1}$ is contained in the timelike bounded region

$$B(a,r) = \{x \in \mathbf{S}_1^{n+1} : -\sinh(r) \leqslant \langle a, x\rangle \leqslant \sinh(r)\} \subset \mathbf{S}_1^{n+1},$$

whose boundaries are two totally umbilical round spheres of radius $\cosh(r)$ and constant jth mean curvature $h_j(r) = \tanh^j(r)$, and the jth mean curvature of M satisfies $|H_j| \geqslant h_j(r)$ when j is odd or $H_j \geqslant h_j(r)$ when j is even, then M must be one of these two round spheres.

Corollary 14. *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space such that its hyperbolic image is contained in a geodesic ball in $\mathbf{H}^{n+1}$ of radius $\varrho > 0$. If there exists a point $p_0 \in M$ where all the principal curvatures $\kappa_i(p_0)$ have the same sign, and the jth mean curvature H_j of M satisfies*

$$|H_j| \geqslant \tanh^j(\varrho), \quad \text{if } j \text{ is odd}$$

or

$$H_j \geqslant \tanh^j(\varrho), \quad \text{if } j \text{ is even},$$

then M must be a totally umbilical round sphere of radius $\cosh(\varrho)$.

Proof. Since there exists a point $p_0 \in M$ where all the principal curvatures have the same sign, we may assume, by choosing the appropriate orientation on M, that $\kappa_i(p_0) < 0$, for $i = 1, \ldots, n$. We will follow now the ideas of Montiel and Ros in [11, Lemma 1] and their use of Gårding inequalities [5] (see also [2]). Actually, taking into account our sign convention in the definition of H_j, $H_j(p_0) > 0$ and $H_j \geqslant \tanh^j(r) > 0$ is positive everywhere on M, so that from the proof of Lemma 1 in [11] it follows that

$$H \geqslant H_j^{1/j} \geqslant \tanh(r) \quad \text{on } M.$$

Thus, Proposition 13 and Corollary 14 automatically follow from our previous Theorem 3 and Theorem 2, respectively.

Remark 15. A particular interesting case for which there always exists a point $p_0 \in M$ where all the principal curvatures $\kappa_i(p_0)$ have the same sign occurs when $\psi(M)$ is contained in the chronological future (or past) of an equator of $\mathbf{S}_1^{n+1}$. For instance, let us assume that the hypersurface $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ is contained in the future of the equator determined by a unit timelike vector $a \in \mathbf{L}^{n+2}$ (the case of the past is similar), and let us orient M by the Gauss map N which is in the same time-orientation as a, that is $\langle a, N\rangle \leqslant -1 < 0$. Since the height function $\langle a, \psi \rangle$ is negative on M, by compactness there exists a point $p_0 \in M$ where it attains its maximum

$$\langle a, \psi(p_0) \rangle = \max_{p \in M} \langle a, \psi(p) \rangle < 0.$$

Therefore, $\nabla \langle a, \psi \rangle (p_0) = a^T(p_0) = 0$ and from (5) it follows that

$$\nabla^2 \langle a, \psi \rangle (p_0)(v, w) = -\langle a, \psi(p_0) \rangle \langle v, w \rangle - \langle a, N(p_0) \rangle \langle A_{p_0}(v), w \rangle \leqslant 0$$

for all $v, w \in T_{p_0}M$. On the other hand, since $\langle a, N \rangle^2 = 1 + \langle a, \psi \rangle^2 + |a^T|^2$ and $a^T(p_0) = 0$, then

$$-\langle a, N(p_0) \rangle = \sqrt{1 + \langle a, \psi(p_0) \rangle^2}.$$

Therefore, choosing $\{e_1, \ldots, e_n\}$ a basis of principal directions at the point p_0 we conclude that

$$\kappa_i(p_0) \leqslant \frac{\langle a, \psi(p_0) \rangle}{\sqrt{1 + \langle a, \psi(p_0) \rangle^2}} < 0$$

for each $i = 1, \ldots, n$.

References

[1] K. Akutagawa, On spacelike hypersurfaces with constant mean curvature in the de Sitter space, *Math. Z.* **196** (1987) 13–19.

[2] J.A. Aledo, L.J. Alías and A. Romero, Integral formulas for compact spacelike hypersurfaces in de Sitter space. Applications to the case of constant higher order mean curvature, *J. Geom. Phys.* **31** (1999) 195–208.

[3] L.J. Alías, On the Ricci curvature of compact spacelike hypersurfaces in de Sitter space, *Geom. Dedicata* **77** (1999) 297–304.

[4] Q-M. Cheng and S. Ishikawa, Spacelike hypersurfaces with constant scalar curvature, *Manuscripta Math.* **95** (1998) 499–505.

[5] L. Gårding, An inequality for hyperbolic polynomials, *J. Math. Mech.* **8** (1959) 957–965.

[6] A.J. Goddard, Some remarks on the existence of spacelike hypersurfaces of constant mean curvature, *Math. Proc. Cambridge Phil. Soc.* **82** (1977) 489-495.

[7] D. Koutroufiotis, Elementary geometric applications of a maximum principle for non-linear elliptic operators, *Arch. Math.* **24** (1973) 97–99.

[8] H. Li, Global rigidity theorems of hypersurfaces, *Ark. Mat.* **35** (1997) 327–351.

[9] S. Markvorsen, A sufficient condition for a compact immersion to be spherical, *Math. Z.* **183** (1983) 407–411.

[10] S. Montiel, An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature, *Indiana Univ. Math. J.* **37** (1988) 909–917.

[11] S. Montiel and A. Ros, Compact hypersurfaces: The Alexandrov theorem for higher order mean curvatures, in B. Lawson and K. Tenenblat, eds., *Differential Geometry*, (Longman, Essex, 1991) 279–296.

[12] V. Oliker, A priori estimates of the principal curvatures of spacelike hypersurfaces in de Sitter space with applications to hypersurfaces in hyperbolic space, *American Math. J.* **114** (1989) 605–626.

[13] J. Ramanathan, Complete spacelike hypersurfaces of constant mean curvature in de Sitter space, *Indiana Univ. Math. J.* **36** (1987) 349–359.

[14] T. Vlachos, A characterization for geodesic spheres in space forms, *Geom. Dedicata* **68** (1997) 73–78.

[15] Y. Zheng, On spacelike hypersurfaces in the de Sitter space, *Ann. Global Anal. Geom.* **13** (1995) 317–321.

[16] Y. Zheng, Spacelike hypersurfaces with constant scalar curvature in the de Sitter spaces, *Differential Geom. Appl.* **6** (1996) 51–54.